MATH 20D Spring 2023 Lecture 22.

The Dirac Delta Function and Laplace Transform.

- The deadline for MATLAB homework 4 has been extended until 11:59pm next Friday.

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- Homework 7 is available and due next Tuesday.

Outline

Contents

Definition

*A body B experiences a force $\mathcal{F}(t)$ over a time window from $[0,\infty)$. The **impulse due to** $\mathcal{F}$ is value of the integral $\int_0^\infty \mathcal{F}(t)dt$.*

- In practice forces only act on bodies for **finite** periods of time. If $\mathcal{F}(t) = 0$ for $t > T$ then we can write the impulse due to $\mathcal{F}$ as

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where m_B is the mass of B and $v(t)$ denote the velocity of B at time t.

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Example

Fix $\varepsilon > 0$ and suppose B experiences a force $\mathcal{F}_\varepsilon$ which is a constant $1/\varepsilon$ Newtons over the time interval $(0,\varepsilon)$ and 0 for $t > \varepsilon$. Calculate the impulse due to $\mathcal{F}_\varepsilon$.

- Consider the force

$$\mathcal{F}_\varepsilon(t) = \begin{cases} 1/\varepsilon, & 0 < t < \varepsilon \\ 0, & t > \varepsilon. \end{cases}$$

So $\mathcal{F}_\varepsilon(t)$ exerts a unit impulse over the time window $(0, \varepsilon)$.

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- Note that if $f\colon [0,\infty) \to \mathbb{R}$ satisfies condition (ii) then $\int_0^\infty f(t)dt = 0$. So rigorously speaking $\delta(t)$ is **not** a function but something called a distribution.

Properties of $\delta(t)$

- Given $f\colon [0,\infty) \to \mathbb{R}$ piecewise continuous we define
$$\int_0^{\infty} f(t)\delta(t)dt = \lim_{\varepsilon \to 0^+} \int f(t)\mathcal{F}_\varepsilon(t)dt.$$

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- If $a \geqslant 0$ is constant we define $\int_0^\infty f(t)\delta(t-a)dt = \lim_{\varepsilon\to 0^+} \int f(t)\mathcal{F}_\varepsilon(t-a)dt$

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Example

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(b) Evaluate $\int_0^\infty e^{-2t}\delta(t-1)dt$.

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(a) Evaluate $\int_0^\infty \sin(3t)\delta(t-\pi/2)dt$.
(b) Evaluate $\int_0^\infty e^{-2t}\delta(t-1)dt$.
(c) Let $a \geqslant 0$ be constant. Show that $\mathscr{L}\{\delta(t-a)\}(s) = e^{-as}$.

Contents

- Suppose $p(t)$ and $q(t)$ are functions. Consider the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = \delta(t), \qquad y(0) = y_0, \quad y'(0) = y_1. \tag{1}$$

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Definition

The solution to the initial value problem (1) is defined as the limit

$$y(t) = \lim_{\varepsilon \to 0^+} y_\varepsilon(t)$$

where $y_\varepsilon(t)$ is the solution to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = \mathcal{F}_\varepsilon(t - a), \qquad y(0) = y_0, \quad y'(0) = y_1.$$

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Example

(a) Using the definition above, solve the initial value problem

$$y'(t) = \delta(t-a), \qquad y'(0) = 0, \qquad t \geqslant 0.$$

(b) Rederive your solution to (a) using the fact that $\mathscr{L}\{\delta(t-a)\}(s) = e^{-as}$.

- As the previous example illustrates, the Laplace transform

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is a useful short-cut for solving initial value problems involving $\delta(t-a)$.

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Example

A mass attached to a spring is released from rest 1m to the right of it's equilibrium. Exactly π second later the mass is struck by a hammer. The system is governed by the symbolic initial value problem

$$y''(t) + 9y(t) = 3\delta(t-\pi); \quad y(0) = 1, \quad y'(0) = 0.$$

Calculate the function $y(t)$.

Definition

Given a differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

the ***impulse response function*** *is the solution to the initial value problem*

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the impulse response function describes the motion of the mass when it struck by a hammer while at rest at the spring's equilibrium.

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Example

Determine the impulse response function associated to the differential equation

$$y'' - 6y' + 13y = g(t)$$